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A FIXED POINT THEOREM IN K-METRIC SPACES

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ABSTRACT

The only difference between ordinary metric and k-metric is in the triangle inequality. In this paper we have shown that instead of this difference a common fixed point theorem for four mappings can be obtained.

Keywords: *Cone metric space, weakly compatible mappings, common fixed point*.

I. INTRODUCTION

In 2012, H. Pajoohesh [1] introduced the concept of k-metric spaces. In this paper we generalize a result of [3] in the language of k-metric spaces. As in [1] a k-metric, where k is a real number ≥ 1, on a nonempty set X is a mapping $d: X \times X \to \mathbb{R}$ such that

(i) $d(x,y) \geq 0 \ \forall \ x,y \in X$,
(ii) $d(x,y) = 0 \Leftrightarrow x = y$,
(iii) $d(x,y) = d(y,x) \ \forall \ x,y \in X$,
(iv) $d(x,y) \leq k(d(x,z) + d(z,y)) \ \forall \ x,y,z \in X$.

The ordered pair (X,d) is called a k-metric space.

Let us consider the mapping $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = (x-y)^2 \ \forall x,y \in \mathbb{R}$. The fact $(a+b)^2 \leq 2(a^2 + b^2) \quad \forall a,b \in \mathbb{R}$ ensures that the mapping d enjoys all the properties of being a k-metric for $k = 2$.

From the definition and the example, just given above, it is clear that every metric is a k-metric ($k = 1$), but a k-metric may not be a metric and every k-metric is an l-metric, where $l \geq k$.

Open balls, closed balls, diameter of non empty sets, open sets (A subset O of a k-metric space (X,d) is said to be open in (X,d) if $\forall \ x \in O \ \exists \ \varepsilon > 0$ such that the open ball $B_d(x,\varepsilon) \subset O$.), closed sets, closure and interior of a set, convergence of a sequence, Cauchy sequence, completeness of k-metric spaces are defined as in case of metric spaces. It is also seen that every k-metric space is first countable and T_4. [2], [4], [5] motivate to work on this field.

II. A COMMON FIXED POINT THEOREM FOR FOUR MAPPINGS

In this section we prove a common fixed point theorem for four self mappings on a complete k-metric space. For that we need following definitions. As in [3]

Definition 1 *Let f and g be self mappings on a set X. A point $x \in X$ is called a* **coincidence point** *of f and g if $fx = gx = w$, where w is called a point of coincidence of f and g.*

Definition 2 *Two self mappings f and g on a set X are said to be* **weakly compatible** *if f and g commute at their coincidence points that is if $fx = gx$ for some $x \in X$, then $fgx = gfx$.*

Theorem 1 *Let (X,d) be a complete k-metric space. Suppose that f, g ,F and G are self mappings on X satisfying the following conditions :*

(a) $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$
(b) $\exists\ \delta > 0, L \geq 0$ satisfying $\delta k + kL(1+k) < 1$ such that
$d(Fx, fy) \leq \delta M(x,y) + Lmin[d(gx,Fx), d(gx,fy)] \quad \forall x,y \in X$, where

$$M(x,y) = max[d(gx,Gy), d(gx,Fx), d(Gy,fy), \frac{1}{2k}\{d(gx,fy) + d(Gy,Fx)\}]$$

(c) $f(X)$ or $g(X)$ is closed.
If $\{f,G\}$ and $\{g,F\}$ are weakly compatible, then f, g, F and G have a unique common fixed point in X.

Proof. Suppose that x_0 is an arbitrary point in X. Since, $f(X) \subseteq g(X)$ and $F(X) \subseteq G(X)$, one may construct a sequence$\{y_n\}$ in X satisfying $y_n = Fx_n = Gx_{n+1}$ and $y_{n+1} = fx_{n+1} = gx_{n+2}$ for all $n \in \mathbb{N} \cup \{0\}$. By the given condition,

$d(Fx_n, fx_{n+1}) \leq \delta M(x_n, x_{n+1}) + Lmin\{d(gx_n, Fx_n), d(gx_{n+1}, fx_{n+1})\}$. Since,

$$\begin{aligned} M(x_n,x_{n+1}) &= max\{d(gx_n,Gx_{n+1}),\ d(gx_n,Fx_n),\ d(Gx_{n+1},fx_{n+1}),\\ &\qquad \frac{1}{2k}[d(gx_n,fx_{n+1}) + d(Gx_{n+1},Fx_n)]\} \\ &= max\{d(y_{n-1},\ y_n), d(y_{n-1},y_n),\ d(y_n,y_{n+1}),\\ &\qquad \frac{1}{2k}[d(y_{n-1},y_{n+1}) + d(y_n,y_n)]\} \\ &\leq max\{d(y_{n-1},y_n),\ d(y_n,y_{n+1}),\ \frac{k}{2k}[d(y_{n-1,y_n}) + d(y_n,y_{n+1})]\} \\ &= max\{d(y_{n-1},y_n),\ d(y_n,y_{n+1}),\ \frac{1}{2}[d(y_{n-1},y_n) + d(y_n,y_{n+1})]\} \end{aligned}$$

and

$$min\{d(gx_n,Fx_n), d(gx_n,fx_{n+1})\} = min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\}$$

we obtain

$$\begin{aligned} d(y_n,y_{n+1}) &= d(Fx_n,fx_{n+1}) \\ &\leq \delta\ \ max\{d(y_{n-1},y_n), d(y_n,y_{n+1})\} \\ &+L\ \ min\{d(y_{n-1},y_n) + d(y_{n-1},y_{n+1})\} \end{aligned}$$

We split-up the proof into the following cases.

Case: 1 If,

$$max\{d(y_{n-1},y_n), d(y_n,y_{n+1})\} = d(y_{n-1},y_n)$$
$$min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\} = d(y_{n-1},y_n)$$

Then

$$\begin{aligned} d(y_n,y_{n+1}) &\leq \delta\ \ d(y_{n-1},y_n) + L\ \ d(y_{n-1},y_n) \\ &= (\delta + L)d(y_{n-1},y_n) \end{aligned}$$

Let $k_1 = (\delta + L)$. Since $\delta k + kL(1+k) < 1$, we have
$k_1 < \frac{1}{k}$ and $d(y_n,y_{n+1}) < k_1 d(y_{n-1},y_n)$

Case: 2 If,

$$max\{d(y_{n-1},y_n),\ \ d(y_n,y_{n+1})\} = d(y_{n-1},y_n)$$
$$min\{d(y_{n-1},y_n), d(y_{n-1},y_{n+1})\} = d(y_{n-1},y_{n+1})$$

Then

$$\begin{aligned} d(y_n,y_{n+1}) &\leq \delta\ \ d(y_{n-1},y_n) + L\ \ d(y_{n-1},y_{n+1}) \\ &\leq \delta d(y_{n-1},y_n) + Lk\ \ d(y_{n-1},y_n) + L\ \ k\ \ d(y_n,y_{n+1}) \\ &\Rightarrow (1 - Lk)d(y_n,y_{n+1}) \leq (\delta + kL)d(y_{n-1},y_n) \\ &\Rightarrow d(y_n,y_{n+1}) \leq \frac{\delta + kL}{1 - kL} d(y_{n-1},y_n) \end{aligned}$$

Let $\frac{\delta+kL}{1-kL} = k_2$, since $\delta k + kL(1+k) < 1$,
$k_2 < \frac{1}{k}$ and $d(y_n,y_{n+1}) \leq k_2 d(y_{n-1},y_n)$

Case: 3 If

$$max\{d(y_{n-1},y_n),d(y_n,y_{n+1})\}=d(y_n,y_{n+1})$$
$$min\{d(y_{n-1},y_n),d(y_{n-1},y_{n+1})\}=d(y_{n-1},y_n)$$

Then

$$d(y_n,y_{n+1})\le\delta\quad d(y_n,y_{n+1})+L\quad d(y_{n-1},y_n)$$
$$\Rightarrow(1-\delta)d(y_n,y_{n+1})\le L\quad d(y_{n-1},y_n)$$
$$\Rightarrow d(y_n,y_{n+1})\le\frac{L}{1-\delta}\quad d(y_{n-1},y_n)$$

Let $\frac{L}{1-\delta}=k_3$, since $\delta k+kL(1+k)<1$,
$k_3<1$ and $d(y_n,y_{n+1})\le k_3 d(y_{n-1},y_n)$.

Case: 4 If

$$max\{d(y_{n-1},y_n),d(y_n,y_{n+1})\}=d(y_n,y_{n+1})$$
$$min\{d(y_{n-1},y_n),d(y_{n-1},y_{n+1})\}=d(y_{n-1},y_{n+1})$$

Then

$$d(y_n,y_{n+1})\le\delta d(y_n,y_{n+1})+Ld(y_{n-1},y_{n+1})$$
$$\le\delta d(y_n,y_{n+1})+Lkd(y_{n-1},y_n)+Lkd(y_n,y_{n+1})$$
$$\Rightarrow(1-\delta-Lk)d(y_{n+1},y_n)\le Lkd(y_{n-1},y_n)$$
$$\Rightarrow d(y_n,y_{n+1})\le\frac{kL}{1-\delta-kL}d(y_n,y_{n-1})$$

Let $\frac{kL}{1-\delta-kL}=k_4$, since $\delta k+kL(1+k)<1$,
$k_4<\frac{1}{k}$ and $d(y_n,y_{n+1})\le k_4 d(y_{n-1},y_n)$
Let $h=max\{k_1,k_2,k_3,k_4\}$ then $h<\frac{1}{k}$ and
$d(y_{n+1},y_n)\le hd(y_n,y_{n-1})\le h^n d(y_0,y_1)$.

Now for $n>m$

$$d(y_m,y_n)\le kd(y_m,y_{m+1})+kd(y_{m+1},y_n)$$
$$\le kd(y_m,y_{m+1})+k^2d(y_{m+1},y_{m+2})+k^2d(y_{m+2},y_n)$$
$$\le kd(y_m,y_{m+1})+k^2d(y_{m+1},y_{m+2})+\ldots+k^{n-m}d(y_{n-1},y_n)$$
$$\le kh^m d(y_0,y_1)+k^2h^{m+1}d(y_0,y_1)+\ldots+k^{n-m}h^{n-1}d(y_0,y_1)$$
$$=h^m(k+k^2h+k^3h^2+\ldots+k^{n-m}h^{n-m-1})d(y_0,y_1)$$
$$=h^m k(1+(kh)+(kh)^2+\ldots+(kh)^{n-m-1})d(y_0,y_1)$$
$$<\frac{h^m k}{1-kh}(\text{since}kh<1)$$
$$\to 0\text{as}m\to\infty$$

Therefore $\{y_n\}$ is a Cauchy sequence in (X,d), since (X,d) is complete, there exist $z\in X$ such that $\lim_{n\to\infty}y_n=z$.
Assume that $g(X)$ is closed, therefore there exist a point $u\in X$ such that $z=gu$
Now we have,

$$d(z,Fu)$$
$$\le k\quad d(z,y_{n+1})+k\quad d(y_{n+1,Fu})$$
$$=k\quad[d(z,y_{n+1})+d(fx_{n+1},Fu)]$$
$$\le k\quad[d(z,y_{n+1})+\delta\quad max\{d(gu,Gx_{n+1}),d(gu,fu),d(Gx_{n+1},fx_{n+1}),$$
$$\frac{1}{2k}[d(gu,fx_{n+1})+d(Gx_{n+1},Fu)]\}+L\quad min\{d(gu,Fu),d(gu,fx_{n+1})\}]$$
$$=k\quad[d(z,y_{n+1})+\delta\quad max\{d(z,y_n),d(z,fu),d(y_n,y_{n+1}),$$
$$\frac{1}{2k}[d(z,y_{n+1})+d(y_n,Fu)]]+L\quad min\{d(z,Fu),d(z,y_{n+1})\}]$$
$$\le k\quad d(z,y_{n+1})+\delta\quad k\quad max\{d(z,y_n),d(z,fu),[k\quad d(y_n,z)+k\quad d(z,y_{n+1})],$$
$$\frac{1}{2k}[d(z,y_{n+1})+k\quad d(y_n,z)+k\quad d(z,Fu)]\}+L\quad min\{d(z,Fu),d(z,y_{n+1})\}$$

Taking limit as $n \to \infty$ we get,

$d(z, fu) \leq \delta k d(z, Fu)$ [by the given condition $\delta k < 1$]

Therefore, $d(z, Fu) = 0 \Rightarrow z = Fu$

Since F and g are weakly compatible, we obtain that,

$gFu = Fgu$

$\Rightarrow gz = Fz$

Since $F(X) \subseteq G(X)$, there exist $v \in X$ such that $Gv = z$

Applying the given condition we get,

$$d(z, fv) = d(Fu, fv)$$
$$\leq \delta \quad max\{d(gu, Gv), d(gu, Fu), d(Gv, fv),$$
$$\frac{1}{2k}[d(gu, fv) + d(Gv, Fu)]\} + L \quad min\{d(gu, Fu), d(gu, fv)\}$$
$$\therefore d(z, fv) \leq \delta d(z, fv)$$
$$\Rightarrow d(z, fv) = 0 \quad (\because \delta k \leq 1)$$
$$\therefore z = fv = Gv$$

Since G and f are weakly compatible, we obtain that

$$fGv = Gfv$$
$$\Rightarrow fz = Gz$$

$$d(Fz, z) = d(Fz, fv)$$
$$\leq \delta max\{d(gz, Gv), d(gz, Fz), d(Gv, fv), \frac{1}{2k}[d(gz, fv) + d(Gv, Fz)]\}$$
$$+ Lmin\{d(gz, Fz), d(Gv, Fz)\}$$
$$= \delta max\{d(Fz, z), d(Fz, Fz), d(z, z), \frac{1}{2k}[d(Fz, z) + d(z, Fz)]\}$$
$$+ Lmin\{d(Fz, Fz), d(Fz, z)\}$$
$$= \delta d(Fz, z)$$
$$\Rightarrow d(Fz, z) = 0$$

So, $gz = Fz = z$. Similarly we get,

$$d(z, fz) = d(Fz, fz)$$
$$\leq \delta max\{d(gz, Gz), d(gz, Fz), d(Gz, fz), \frac{1}{2k}[d(gz, fz) + d(Gz, Fz)]\} + Lmin\{d(gz, Fz), d(gz, fz)\}$$
$$= \delta max\{d(z, fz), d(z, z), d(fz, fz), \frac{1}{2k}[d(z, fz) + d(fz, z)]\}$$
$$+ Lmin\{d(z, z), d(z, fz)\}$$
$$= \delta d(z, fz)$$
$$\Rightarrow d(z, fz) = 0$$

So, $Gz = fz = z$ and therefore z is common fixed point of f, g, F and G.

Uniqueness of such common fixed point:

Let $p \in X$ be also a common fixed point of f, g, F and G. Again applying the given condition we get,

$$d(z, p) = d(Fz, fp)$$
$$\leq \delta max\{d(gz, Gp), d(gz, Fz), d(Gp, fp), \frac{1}{2k}[d(gz, fp) + d(Gp, Fz)]\}$$
$$+ Lmin\{d(gz, Fz), d(gz, fp)\}$$
$$= \delta max\{d(z, p), d(z, z), d(p, p), \frac{1}{2k}[d(z, p) + d(p, z)]\}$$
$$+ Lmin\{d(z, z), d(z, p)\}$$
$$= \delta d(z, p)$$

i.e. $d(z, p) \leq \delta d(z, p)$

This implies that $d(z, p) = 0$ and so $z = p$

Hence f, g, F and G have a unique common fixed point in X.

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